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THE CLIQUE COMPLEX AND HYPERGRAPH MATCHING

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The *width* of a hypergraph $\mathcal{F}$ is the minimal $t = w(\mathcal{F})$ for which there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{F}$, $F_i \cap F \neq \emptyset$ for some $1 \le i \le t$. The *matching width* of $\mathcal{F}$ is the minimal $t = mw(\mathcal{F})$ such that for any matching $\mathcal{M} \subset \mathcal{F}$ there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{M}$, $F_i \cap F \neq \emptyset$ for some $1 \le i \le t$. The following extension of the Aharoni-Haxell matching Theorem [3] is proved:

Let $\{\mathcal{F}_i\}_{i=1}^m$ be a family of hypergraphs such that for each $\emptyset \neq I \subset [m]$ either $mw(\cup_{i \in I} \mathcal{F}_i) \ge |I|$ or $w(\cup_{i \in I} \mathcal{F}_i) \ge 2|I| - 1$, then there exists a matching $F_1, \ldots, F_m$ such that $F_i \in \mathcal{F}_i$ for all $1 \le i \le m$.

This is a consequence of a more general result on colored cliques in graphs. The proofs are topological and use the Nerve Theorem.

1. Introduction

Let $\mathcal{F} \subset 2^X$ be a hypergraph on a finite ground set X. The *width* of $\mathcal{F}$ is the minimal $t = w(\mathcal{F})$ for which there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{F}$, $F_i \cap F \neq \emptyset$ for some $1 \le i \le t$.

A *matching* in $\mathcal{F}$ is a subhypergraph $\mathcal{M} \subset \mathcal{F}$ such that $F \cap F' = \emptyset$ for all $F \neq F' \in \mathcal{M}$. The *matching width* of $\mathcal{F}$ is the minimal $t = mw(\mathcal{F})$ such that for any matching $\mathcal{M} \subset \mathcal{F}$ there exist $F_1, \ldots, F_t \in \mathcal{F}$ such that for any $F \in \mathcal{M}$, $F_i \cap F \neq \emptyset$ for some $1 \le i \le t$.

Let $\{\mathcal{F}_i\}_{i=1}^m$ be a family of hypergraphs. A *system of disjoint representatives (SDR)* of $\{\mathcal{F}_i\}_{i=1}^m$ is a matching $F_1, \ldots, F_m$ such that $F_i \in \mathcal{F}_i$ for

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$1 \leq i \leq m$. Aharoni and Haxell [3] have recently proved the following remarkable extension of Hall's matching theorem:

Theorem 1.1. *If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $mw(\cup_{i\in I}\mathcal{F}_i) \geq |I|$ for all $I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Another closely related result of Aharoni and Haxell [4] is the following

Theorem 1.2. *If $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w(\cup_{i\in I}\mathcal{F}_i) \geq 2|I|-1$ for all $I \subset [m]$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

A common generalization of theorems 1.1 and 1.2 was obtained by Aharoni and Chudnovsky [2]:

Theorem 1.3. *Let $1 \leq d \leq m$. Suppose $\{\mathcal{F}_i\}_{i=1}^m$ satisfies $w(\cup_{i\in I}\mathcal{F}_i) \geq 2|I|-1$ if $|I| \leq d$, and $mw(\cup_{i\in I}\mathcal{F}_i) \geq |I|$ if $|I| > d$. Then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Theorems 1.1 and 1.3 are proved using the Sperner Lemma and rely on the existence of special triangulations of the simplex. Aharoni [1] noted that the method of [3] shows that if for any $\emptyset \neq I \subset [m]$ the matching complex of $\cup_{i\in I}\mathcal{F}_i$ is $(|I|-2)$-connected then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.

In this note we use another topological approach to prove the following extension of Theorem 1.3 conjectured in [2].

Theorem 1.4. *Let $\{\mathcal{F}_i\}_{i=1}^m$ be a family of hypergraphs such that for each $\emptyset \neq I \subset [m]$ either $mw(\cup_{i\in I}\mathcal{F}_i) \geq |I|$ or $w(\cup_{i\in I}\mathcal{F}_i) \geq 2|I|-1$, then $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR.*

Theorem 1.4 is a special case of a result on colored cliques in graphs. Let $G=(V,E)$ be a simple graph. Let $\Gamma(v)$ denote the set of neighbors of a vertex $v \in V$, and $G[A]$ the induced graph on a subset of vertices $A \subset V$. The join of two simplicial complexes K and L is denoted by $K * L$. For $0 \leq i \leq t$ let K_i consist of two isolated points a_i and a_i', then $S^t = K_0 * \ldots * K_t$ is the octahedral $t-$dimensional sphere. The $1-$dimensional skeleton of S^t is denoted by $(S^t)^{(1)}$.

A graph $G=(V,E)$ *has property* $P(l,t)$ if there exists an $A \subset V$ such that:
(i) $\cap_{i=1}^l \Gamma(v_i) \cap A \neq \emptyset$ for any $v_1, \ldots, v_l \in V$.
(ii) $G[A]$ does not contain $(S^t)^{(1)}$ as an induced subgraph.

We shall also say that G *satisfies* $P(l,t)$ *with respect to* A.

Remarks.
(a) We note two choices of parameters which will be relevant to Theorem 1.4: If G satisfies $P(l,0)$ with respect to A, then A does not contain an induced S^0 i.e. A is a clique. On the other hand, G satisfies property $P(l,\infty)$ iff $\cap_{i=1}^l \Gamma(v_i) \neq \emptyset$ for any $v_1, \ldots, v_l \in V$.

(b) Let $\mathcal{F} \subset 2^X$ be a hypergraph, possibly with multiple edges. The *disjointness graph* $G_{\mathcal{F}} = (V, E)$ associated with $\mathcal{F}$ has vertex set $V = \mathcal{F}$ and $\{F, F'\} \subset \mathcal{F}$ forms an edge if $F \cap F' = \emptyset$. Since a matching in $\mathcal{F}$ corresponds to a clique in $G_{\mathcal{F}}$ it follows that $mw(\mathcal{F}) > r$ iff $G_{\mathcal{F}}$ satisfies $P(r, 0)$. Similarly $w(\mathcal{F}) > r$ iff $G_{\mathcal{F}}$ satisfies $P(r, \infty)$.

Let $V = \cup_{i=1}^m V_i$ be a partition of the vertices of a graph G into non-empty sets. A clique $\sigma \subset V$ is *colored* if $\sigma \cap V_i \neq \emptyset$ for all $1 \leq i \leq m$. For integers $l \geq 1$ and $t \geq 0$ let $\kappa(l,t) = \max\{l - t, \lfloor \frac{l}{2} \rfloor\}$. Our main result is a sufficient condition for the existence of colored cliques in G:

Theorem 1.5. *If for any $\emptyset \neq I \subset [m]$ there exist l_I, t_I such that $G[\cup_{i\in I} V_i]$ satisfies $P(l_I, t_I)$ and $\kappa(l_I, t_I) \geq |I| - 1$ then G contains a colored clique.*

Theorem 1.5 ⇒ Theorem 1.4. Let $\mathcal{F}$ denote the disjoint union of the $\mathcal{F}_i$'s, and let $G_{\mathcal{F}} = (V, E)$ with the partition $V = \cup_{i=1}^m V_i$ where $V_i = \mathcal{F}_i$. If $\emptyset \neq I \subset [m]$ then either $mw(\cup_{i\in I} \mathcal{F}_i) \geq |I|$ or $w(\cup_{i\in I} \mathcal{F}_i) \geq 2|I| - 1$. It follows from the remarks above that $G[\cup_{i\in I} V_i]$ satisfies $P(l, t)$ where (l, t) is either $(|I| - 1, 0)$ or $(2|I| - 2, \infty)$. Since $\kappa(l, t) = |I| - 1$ in either case, it follows by Theorem 1.5 that G contains a colored clique. Therefore $\{\mathcal{F}_i\}_{i=1}^m$ has an SDR. ∎

The proof of Theorem 1.5 is topological and uses the clique complex of a graph. In Section 2 we discuss a version of the Nerve Theorem which is our main technical tool. In Section 3 we relate property $P(l, t)$ to the homology of the clique complex. The proof of Theorem 1.5 is completed in Section 4.

2. The Nerve Theorem

For a finite simplicial complex X, let $H_j(X)$ ($\tilde{H}_j(X)$) denote the (reduced) j−th simplicial homology of X with coefficients in some fixed field $\mathbb{F}$. The j−dimensional skeleton of X is denoted by $X^{(j)}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a finite family of subcomplexes of X such that $\cup_{i\in I} U_i = X$. For $\sigma \subset I$ let $U_\sigma = \cap_{i\in\sigma} U_i$. The *nerve* of $\mathcal{U}$ is the simplicial complex $N = N(\mathcal{U})$ on the vertex set I whose simplices are all $\sigma \subset I$ such that $U_\sigma \neq \emptyset$. The basic form of Leray's Nerve Theorem asserts that if U_σ is contractible for any $\sigma \in N$ then X is homotopy equivalent to N (see also [6]). For more refined versions and their combinatorial applications see Björner's survey [5]. We shall need the following homology variant of the Nerve Theorem. For completeness we include a short proof based on a standard application of the Leray spectral sequence (see e.g. [7]).

Theorem 2.1. *If $\tilde{H}_j(U_\sigma) = 0$ for all $\sigma \in N^{(k)}$ and $0 \leq j \leq k - \dim\sigma$, then*

(i) $\tilde{H}_j(X)\cong\tilde{H}_j(N)$ *for* $0\le j\le k$.
(ii) *If* $H_{k+1}(N)\neq 0$ *then* $H_{k+1}(X)\neq 0$.

Proof. Let $\{E_r\}$ denote the cohomology spectral sequence of the cover $\mathcal{U}$ and let $C^*(N)$ be the cochain complex of the nerve N. Let $\varphi_p : C^p(N)\to \oplus_{\sigma\in N^{(p)}} H^0(U_\sigma)=E_1^{p,0}$ denote the natural diagonal injection. The assumption $\tilde{H}^0(U_\sigma)=0$ whenever $\dim\sigma\le k$ implies that φ_p is an isomorphism for $0\le p\le k$. It follows that $E_2^{p,0}\cong H^p(N)$ for $0\le p\le k$. Since $E_1^{p,q}=\oplus_{\sigma\in N^{(p)}} H^q(U_\sigma)=0$ for $q\ge 1$ and $p+q\le k$ we conclude that $H^p(X)\cong E_2^{p,0}\cong H^p(N)$ for $0\le p\le k$ and $E_\infty^{k+1,0}=E_2^{k+1,0}$.

To prove (ii) note that since φ_k is an isomorphism, it follows that φ_{k+1}^*: $H^{k+1}(N)\to E_2^{k+1,0}$ is injective. Therefore $H^{k+1}(N)\neq 0$ implies $E_\infty^{k+1,0}=E_2^{k+1,0}\neq 0$ and so $H^{k+1}(X)\neq 0$. ∎

3. Homology of the Clique Complex

The *Clique Complex* of a graph $G=(V,E)$ is the simplicial complex $X(G)$ on the vertex set V whose simplices are all cliques $\sigma\subset V$.

Proposition 3.1. *Let* $l\ge 1$ *and* $t\ge 0$. *If* G *satisfies* $P(l,t)$ *then* $\tilde{H}_j(X(G))=0$ *for* $0\le j\le\kappa(l,t)-1$.

Proof. For $v\in V$ let $\Gamma'(v)=\Gamma(v)\cup\{v\}$ and $U_v=X(G[\Gamma'(v)])$. Suppose G satisfies $P(l,t)$ with respect to $A\subset V$ and let $Z=\cup_{v\in A}U_v\subset X(G)$. If $\tau=\{u_0,\ldots,u_r\}\in X(G)$ and $r\le l-1$ then there exists a $v\in\cap_{i=0}^r\Gamma(u_i)\cap A$, hence $\tau\subset U_v$. It follows that $Z\supset X(G)^{(l-1)}\supset X(G)^{(\kappa(l,t)-1)}$. It therefore suffices to show that $\tilde{H}_j(Z)=0$ for $0\le j\le\kappa(l,t)-1$.

Let N denote the nerve of the cover $\{U_v\}_{v\in A}$ of Z. Note that if $\sigma=\{v_0,\ldots v_p\}\subset A$ forms a clique in G then any clique τ in $G[\cap_{i=0}^p\Gamma'(v_i)]=G'$ is contained in the clique $\sigma\cup\tau\subset G'$, hence $U_\sigma=X(G')$ is a cone over σ and is therefore contractible.

Since G does not contain an induced $(S^t)^{(1)}$ for $t\ge|V|/2$ it suffices to prove the proposition for finite t. We apply induction on t. If $t=0$ then A is a clique, hence by the preceding remark U_σ is non-empty and contractible for any $\sigma\subset A$. It follows that N is the simplex on A hence by Theorem 2.1 $\tilde{H}_*(Z)\cong\tilde{H}_*(N)=0$.

Suppose $t\ge 1$. Then $\kappa(l,t)\le l-1$. Hence if $0\le p\le\kappa(l,t)$ and $\sigma=\{v_0,\ldots,v_p\}\subset A$ then $\cap_{i=0}^p\Gamma'(v_i)\cap A\neq\emptyset$. It follows that $U_\sigma=X(G[\cap_{i=0}^p\Gamma'(v_i)])\neq\emptyset$, therefore N contains the $\kappa(l,t)$-th skeleton of the simplex on A and so $\tilde{H}_j(N)=0$ for $0\le j\le\kappa(l,t)-1$.

In order to show $\tilde{H}_j(Z)=0$ for $0\le j\le \kappa(l,t)-1$, it therefore suffices, by Theorem 2.1, to verify that the cover $\{U_v\}_{v\in A}$ of Z satisfies $\tilde{H}_j(U_\sigma)=0$ for any $0\le p\le\kappa(l,t)-1$, $\sigma=\{v_0,\dots,v_p\}\subset A$ and $0\le j\le\kappa(l,t)-1-p$.

If σ is a clique then U_σ is contractible and we are done. Otherwise $p\ge 1$ and there exist two vertices, say v_0 and v_1, such that $\{v_0,v_1\}\notin E$. Let $V'=\cap_{i=0}^{p}\Gamma'(v_i)$, $G'=G[V']$ and $A'=A\cap V'$. Clearly $G[A']$ does not contain an induced $(S^{t-1})^{(1)}$, and for any $l-p-1$ vertices $u_1,\dots,u_{l-p-1}\in V'$, $\cap_{j=1}^{l-p-1}\Gamma(u_j)\ \bigcap\ A'\supset\cap_{j=1}^{l-p-1}\Gamma(u_j)\ \bigcap\ \cap_{i=0}^{p}\Gamma(v_i)\ \bigcap\ A\neq\emptyset$. It follows that G' satisfies $P(l-p-1,t-1)$ with respect to A'. Therefore by induction $\tilde{H}_j(U_\sigma)=\tilde{H}_j(X(G'))=0$ for $0\le j\le\kappa(l-p-1,t-1)-1$. Now $\kappa(l-p-1,t-1)=\max\{l-t-p,\lfloor\frac{l-p-1}{2}\rfloor\}\ge\kappa(l,t)-p$, hence $\tilde{H}_j(U_\sigma)=0$ for $0\le j\le\kappa(l,t)-1-p$. ∎

Remarks.
a) The following examples show that Proposition 3.1 is in a sense sharp. Let $0\le t\le k$ and let $V=\{a_i,a_i'\}_{i=0}^{k-1}\cup\{b_i,b_i'\}_{i=0}^{t-1}$. For $0\le i\le t-1$ let K_i denote the (1-dimensional) simplicial complex consisting of the two edges $\{a_i,b_i\}$ and $\{a_i',b_i'\}$. For $t\le i\le k-1$ let K_i consist of the two isolated points a_i and a_i'. Consider the join $K=K_0*\dots*K_{k-1}$ and let $G=(V,E)=K^{(1)}$ be its 1-dimensional skeleton. Let $A=\{a_i,a_i',b_i,b_i'\}_{i=0}^{t-1}\cup\{a_i\}_{i=t}^{k-1}$.

Claim 3.2. *G satisfies $P(k+t-1,t)$ with respect to A.*

Proof. We first show that if $C\subset V$ with $|C|=k+t-1$, then $\cap_{c\in C}\Gamma(c)\cap A\neq\emptyset$. Indeed, if $|C\cap\{a_i,a_i',b_i,b_i'\}|\le 1$ for some $0\le i\le t-1$, e.g. $C\cap\{a_i,a_i',b_i\}=\emptyset$, then $a_i'\in\cap_{c\in C}\Gamma(c)\cap A$. Otherwise $|C\cap\{a_i,a_i',b_i,b_i'\}|\ge 2$ for all $0\le i\le t-1$ hence $|C\cap\{a_i,a_i'\}_{i=t}^{k-1}|\le|C|-2t=k-t-1$. It follows that $C\cap\{a_i,a_i'\}=\emptyset$ for some $t\le i\le k-1$, hence $a_i\in\cap_{c\in C}\Gamma(c)\cap A$.

We next show that $G[A]$ does not contain an induced octahedral t-sphere. Assume to the contrary that $C\subset A$ satisfies $|C|=2t{+}2$ and $G[C]\cong(S^t)^{(1)}$. Let $C_0=C\cap(\{a_i,b_i\}_{i=0}^{t-1}\cup\{a_i\}_{i=t}^{k-1})$ and $C_1=C\cap(\{a_i',b_i'\}_{i=0}^{t-1}\cup\{a_i\}_{i=t}^{k-1})$. Both C_0 and C_1 are cliques in $G[C]$, hence $|C_0|,|C_1|\le t{+}1$. Since $2t{+}2=|C|=|C_0\cup C_1|$ it follows that $|C_0|=|C_1|=t{+}1$, $C_0\cap C_1=\emptyset$ and $C\cap\{a_i\}_{i=t}^{k-1}=\emptyset$. Therefore there exists an $0\le i\le t-1$ such that $C_0\supset\{a_i,b_i\}$. But this is a contradiction since an octahedral sphere does not contain an edge $\{u,v\}$ with $\Gamma'(u)=\Gamma'(v)$. ∎

Now $X(G)=K$ is homotopic to S^{k-1} hence $\tilde{H}_{k-1}(X(G))\neq 0$. It follows that the assumption $P(k+t,t)$ in Proposition 3.1 cannot be replaced by $P(k+t-1,t)$ when $0\le t\le k$.
b) The following result can be proved along the lines of Proposition 3.1 by applying a relative version of the Nerve Theorem.

Proposition 3.3. *If G satisfies $P(2k-1,\infty)$ with respect to $A \subset V$ then $H_j(X(G),X(G[A])) = 0$ for $0 \le j \le k-1$. In particular $H_{k-1}(X(G[A])) = 0$ implies $\tilde{H}_j(X(G)) = 0$ for $0 \le j \le k-1$.* ∎

4. Colored cliques

Proof of Theorem 1.5. By assumption $G = G[\cup_{i=1}^m V_i]$ satisfies $P(l,t)$ for some pair (l,t) such that $\kappa(l,t) \ge m-1$. Hence $H_{m-2}(X(G)) = 0$ by Proposition 3.1. For $1 \le i \le m$ let $W_i = \cup_{j \ne i} V_j$ and $Y_i = X(G[W_i])$. If G contains no clique $v_1,\dots,v_m$ with $v_i \in V_i$ then $X(G) = \cup_{i=1}^m Y_i$. Let $0 \le p \le m-2$, then for any $\sigma = \{i_0,\dots,i_p\} \subset [m]$ there exists a pair (l,t) (depending on σ) such that $G[\cup_{j \notin \sigma} V_j]$ satisfies $P(l,t)$ and $\kappa(l,t) \ge m-p-2$. Applying Proposition 3.1 it follows that $Y_\sigma = \cap_{i\in\sigma} Y_i = X(G[\cup_{j\notin\sigma} V_j])$ satisfies $\tilde{H}_j(Y_\sigma) = 0$ for $0 \le j \le m-3-\dim\sigma$. The cover $\mathcal{Y} = \{Y_i\}_{i=1}^m$ of $X(G)$ therefore meets the conditions of Theorem 2.1 with $k = m-3$. Since $H_{m-2}(X) = 0$ it follows by 2.1(ii) that $H_{m-2}(N(\mathcal{Y})) = 0$. But $N(\mathcal{Y})$ is clearly the $(m-2)$-skeleton of the $(m-1)$-simplex on the vertex set $[m]$, hence $H_{m-2}(N(\mathcal{Y})) \cong \mathbb{F}$, a contradiction. ∎

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